

Extraction of nucleon momentum distributions from inclusive electron scattering on nuclei

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Abstract

We address the problem of extracting single-nucleon momentum distributions $n(p)$ from inclusive electron scattering data. A model for these relates nuclear and nucleon structure functions (SF) through an intermediate SF f^{PN} for a nucleus of point-particles. In addition to the asymptotic limit (AL) which depends on $n(p)$, f^{PN} contains generally q -dependent Final State Interactions (FSI) parts. In the inverse problem one wishes to separate q -dependent FSI from the AL. In general it suffices to know the structure of the theory, but not numerical results. It appears, that in the q -range of the analyzed electron scattering data, FSI are only weakly q -dependent, making it virtually impossible to obtain parameters in a free fit of the parametrized components of f . Imposing a restriction, we obtain $n(p)$ for Fe and ${}^4\text{He}$.

The simplest bulk properties of a many-body system in its ground state are the number (charge) and momentum distributions of the constituents, or of their centers if the latter are composite. Their obvious importance invites measurements, allowing the extraction of these observables in a model-independent fashion. This appears feasible for nuclear charge distributions, using precision data on elastic electron scattering and on muonic atoms¹. The situation is different for single-nucleon momentum distribution (MD) $n(p)$. Contrary to the case of charge densities, attempts to determine $n(p)$ have met with difficulties, which are frequently circumvented by calling in theoretical MD²⁻⁵.

We illustrate the above for exclusive $A(e, e'p)(A-1)_n$ and semi-inclusive $A(e, e'p)X$ reactions. An analysis usually starts with the Plane Wave Impulse Approximation (PWIA) which expresses yields in terms of single-hole spectral functions. For the two types of reactions these functions are in turn respectively linked to the occupation probability n_α of a single-particle orbit α and the total momentum distribution $n(p)$ ⁶. However, the PWIA result is distorted by Final State interaction (FSI) which as a rule cannot be removed experimentally. An outstanding exception is the case of near-elastic, high- q electro-dissociation of the D , which produced $n^D(p)$ ⁷.

A similar unsatisfactory situation holds for totally inclusive processes $A(e, e')X$, which may also be treated in the PWIA and also require the calculation of FSI^{8,9}. Similar problems beset indirect attempts to obtain MD. These are based on extracted scaling functions $\xi(q, y)$ which depend on the momentum transfer q and some scaling variable y . In particular Gersch-Rodriguez-Smith (GRS)-type of theories¹⁰ directly relate $\lim_{q \rightarrow \infty} \xi(q, y \leq 0)$ to the MD. Plots of $\xi(q, y \leq 0)$ as function of y display coarse-grained scaling, i.e. clustering of data points for different q as function of y . Perfect scaling is interpreted as the absence of FSI, in principle enabling the extraction of $n(p)$, while imperfect scaling manifests FSI¹¹⁻¹⁴. The latter is again difficult to remove experimentally.

In an alternatively approach one searches for a plateau in $\xi(q, \langle y \rangle \leq 0)$ for binned $\langle y \rangle$ as function of q or of the squared 4-momentum $Q = q^2 - \nu^2$ and which is associated with the asymptotic limit (AL) for that y , and thus with $n(y)$ ^{12,15,16}. Both approaches assume that

FSI terms are recognizable by a clear Q^2 -signature.

We now proceed and present a different method which proved eminently succesful in the extraction of MD from data on the SF (or response) $\phi(q, y)$ of liquid ^4He ^{17–19} and Ne ²⁰. For systems with smooth inter-particle interactions V , SF may be expanded in a series in $1/q$ ¹⁰

$$\phi(q, y) = \sum_{n \geq 0} \left(\frac{M}{q} \right)^n F_n(y) = \phi^{as}(y) + \phi^{odd}(q, y) + \Delta\phi^{even}(q, y) \quad (1a)$$

$$F_0(y) = \lim_{q \rightarrow \infty} \phi(q, y) = \frac{1}{4\pi^2} \int_y^\infty dp p n(p), \quad (1b)$$

$$\phi^{odd}(q, y) = \sum_{n, odd} \left(\frac{M}{q} \right)^n F_n(y) = U^{(o)}(q) y \sum_n a_n^{(o)} y^{2n} \exp(-[A^{(o)} y]^2) \quad (1c)$$

$$\phi^{even}(q, y) = \sum_{n, even} \left(\frac{M}{q} \right)^n F_n(y) = U^{(e)}(q) \sum_n a_n^{(e)} y^{2n} \exp(-[A^{(e)} y]^2) \quad (1d)$$

The above decomposition is in terms of coefficient functions $F_n(y)$, even and odd in y , which are preceded by factors with, respectively, even and odd powers of $1/q$. The even AL, Eq. (1a), depends in a simple fashion on the MD.

Eqs. (1) instruct how to calculate $\phi(q, y)$ from $F_n(q, y)$, which requires as input $n(p)$, V and special density matrices¹⁰. In practice this is feasible for the AL and for the dominant FSI coefficients $F_1(y), F_2(y)$ ¹⁹. Alternatively one may wish to use SF data, in order to conversely extract $n(p)$. Eq. (1a) then serves to express information on the underlying dynamics, which one incorporates in parametrizations as in (1c), (1d). Algorithms show how to separate q -independent AL from, in principle q -dependent, FSI without knowledge of computed F_n . From fits, including those for the AL, one generates a MD, which appears to be in good agreement with accurate calculations^{17–19}.

Two dynamical extensions complicate the above NR approach. The first one arises if the bare inter-constituent interaction is not smooth or even singular, which requires the replacement of V by $V_{eff} = t_q$, the t -matrix associated with V . Since V_{eff} depends on q , the GRS q -signature of FSI as in (1c), (1d) will be modified, although for quantum gases not drastically¹⁹.

A second complication occurs for molecular systems with additional degrees of freedom,

which may be excited along with the motion of the molecular centers of mass. Prime examples are rotations and vibrations in a gas of H_2 molecules. Again, because one knows the required additional dynamics for NR systems, the above complications can be handled, both in calculations of the response and in methods to extract $n(p)$. As long as FSI parts possess a distinct q -dependence, the algorithms remain applicable^{21,22}.

We now turn to inclusive electron scattering from nuclei, for which the cross section per nucleon reads

$$\frac{d^2\sigma_{eA}(E; \theta, \nu)/A}{d\Omega d\nu} = \frac{2}{A}\sigma_M(E; \theta, \nu) \left[\frac{xM^2}{Q^2} F_2^A(x, Q^2) + \tan^2(\theta/2) F_1^A(x, Q^2) \right], \quad (2)$$

The inclusive, as well as the Mott cross section σ_M , are usually measured as functions of E, θ, ν which are the beam energy, scattering angle and energy loss. The above SF $F_{1,2}^A(x, Q^2)$ describe the scattering of unpolarized electrons from non-oriented targets and are conventionally expressed as functions of the squared 4-momentum $Q^2 = \mathbf{q}^2 - \nu^2$ and the Bjorken variable $x = Q^2/2M\nu$ with range $0 \leq x \leq A$. The strong variation in both cross sections in (2) can be exploited by using the tempered, dimensionless ratio

$$h_A(E; \nu, \theta) \equiv \frac{M}{2} \left(\frac{d^2\sigma_{eA}(E; \nu, \theta)/A}{\sigma_M(E; \nu, \theta)} \right) = \frac{xM^2}{Q^2} F_2^A(x, Q^2) + \tan^2(\theta/2) F_1^A(x, Q^2) \quad (3)$$

In spite of technical complications, due to a finite number of constituents which obey Fermi statistics, a description of inclusive scattering of relatively low-energy leptons is not basically different from the above NR case. In particular one may still venture to use the notion of a potential for the description of NN collisions in FSI at medium $q_{lab} \approx q \lesssim 0.5$ GeV. However, data for the above kinematics yield only information on MD for restricted p . In order to extend that range, one needs considerably larger momentum and energy transfers q, ν . Those are provided by multi-GeV beam energies, which may excite sub-nucleonic degrees of freedom.

Contrary to NR systems, there is no way to accurately compute SF for nuclei with composite constituents. Whatever the approach, it seems evident that with nucleons as major constituents, one has to relate nuclear SF F_k^A in (2) to those of the nucleon F_k^N , which

in practice is the p, n -weighted $F_k^{p,n}$ close to $F_k^D/2$. We shall use below such a previously formulated relation, which refers only to nucleonic and sub-nucleonic degrees of freedom²³. It specifically disregards virtual cloud pions (see ref. 24¹).

$$F_k^A(x, Q^2) = \int_x^A \frac{dz}{z^{2-k}} f^{PN}(z, Q^2) F_k^N\left(\frac{x}{z}, Q^2\right) \quad (4)$$

The above contains SF for a nucleon F_k^N and f^{PN} for a nucleus of point-particles, which has to be computed. Its asymptotic limit depends on the MD of point-nucleons (or of the centers of composite ones) and on FSI parts, which account for the distribution of the energy-momentum transfer to several core nucleons through NN collisions. In the kinematic range of interest those occur at relativistic momenta q , for which the notion of a local, energy-independent potential breaks down. Again one needs $V_{eff} = t(q)$, which in general is off-shell, yet can be parametrized in terms of observable NN scattering data, and which permits a calculation of the analog of the SF $\phi(q, y)^{13,28}$. As a first step one replaces there the NR GRS-West scaling variable by a relativistic one²⁹

$$y = \frac{M\nu}{q} \left(1 - \frac{q^2}{2M\nu}\right) \rightarrow y_G \approx \frac{M\nu}{q} \left(1 - \frac{Q^2}{2M\nu}\right) = \frac{M}{\sqrt{1 + 4M^2x^2/Q^2}} (1 - x) \quad (5)$$

Next one establishes the following transition from the SF $\phi(q, y_G)$ to a relativistic analog²⁸

$$f^{PN}(x, Q^2) = MD(x, Q^2) \phi[q(x, Q^2), y_G(x, Q^2)], \quad (6)$$

where D is a kinematic factor, which guarantees proper normalization of f^{PN} . The theory^{13,28} based on (6) accounts well for the data^{15,30}

¹Eq. (4) is easily interpreted in terms of momentum fractions (MF): The MF of a quark in a nucleus is the product of MF's of a quark in a nucleon and of a nucleon in a nucleus. SF are the probabilities of their occurrence. The $Q^2 \rightarrow \infty$ limit of a MF equals the observable Bjorken variable $x = Q^2/2M\nu$. Eq. (4) conjectures a similar relation for large finite Q^2 . Similar expressions have been derived in perturbation theory²⁵ with the nucleon off its mass-shell. F_k^N in (4) relates to free nucleons.

We now consider the inverse problem, i.e. the extraction of MD from inclusive scattering data. We choose the recent Fe data for $E = 4.05$ GeV¹⁵ and older ^4He data for $E \leq 3.6$ GeV³⁰ and determine the reduced cross sections h_A , Eq. (3). Its right hand side is expressed by means of the nuclear SF (4), which the nucleon SF F_k^N . The latter can be decomposed as $F^N = F^{N,NI}(x, Q^2) + \delta(1-x)\tilde{F}^N(Q^2)$, containing the nucleon inelastic (NI) and nucleon elastic (NE) parts. The former have been parametrized^{26,27}, whereas the latter are combinations of known static form factors.

Eq. (6) relates f^{PN} to $\phi(q, y_G)$, with parametrizations (1a) as in the NR case. In particular its AL (1b) is generated by a MD, for which we choose a sum of two centered gaussians³¹

$$n(p, \gamma_k) = \frac{n(0)}{1 + \epsilon} \left(\exp[-(p/p_1)^2] + \epsilon \exp[-(p/p_2)^2] \right) \quad (7)$$

In initial runs we found it impossible to obtain a free fit for the FSI part and decided to generate starting values for parameters, by fitting the theoretical FSI to (1c), (1d). In the relevant q -range $\phi^{th}(q, y_G)$ in (1a) appears to have the following striking features:

i) ϕ^{odd} is negligibly small, while the remnant even FSI part $\Delta\phi^{even}$ may reach 35% of $F_0(y_G)$ for $|y_G| \approx 0.2$ GeV, and up to 15% for $|y_G| \approx 0.4$ GeV.

ii) ϕ^{odd} and $\Delta\phi^{even}$ are only weakly q -dependent.

Both features are properties of an effective interaction of a diffractive nature, for instance generated by a nearly imaginary $V_{eff} = t_q$ which interchanges the roles of odd and even parts in y_G . In the relevant range, its strength $\text{Im} t_q \propto \sigma_q^{tot}$ happens to be hardly q -dependent, which spells difficulties for the separation of the AL from FSI with that property.¹ Additional evidence may be implicit in an analysis by Ciofi *et al*³⁵ who found that extracted q -dependent

¹Not only NN interactions produce a diffractive σ_q^{el} . For example, the atom-atom interaction in liquid ^4He has a strong, short-range repulsion³². This causes damped oscillation in σ_q^{tot} for increasing q . However for all, but the largest relevant q , there still is a discernible q -dependence³³. The latter property and the relative weakness of FSI, enable the extraction of the MD from the

scaling functions can be fitted by a q -independent parametrization. This surely is compatible with weakly q -dependent FSI.

We return to attempts to extract $n(p)$ from Fe data, which we divided in 10 $\langle q \rangle$ bins. The lowest two, which correspond to $\theta = 15, 23^\circ$, resist any fit. This observation is in line with the validity of the relation (4), which predicts deterioration with decreasing Q^2 . For increasing Q^2 , data in a given $\langle q \rangle$ bin often produce excellent fits for h_{Fe} with relative deviations from data, rarely exceeding 5%, and usually staying under 2%! Nevertheless, the corresponding parameters occasionally appear far from their starting values.

The failure of a free fit forces the imposition of a restriction, e.g. a given central value $n(0)$ in the neighbourhood of the theoretical one. Such a choice leads to a well-determined, reasonable value for p_1 , the width of the dominant gaussian in the MD (7). In contrast, the width of the second gaussian appears strongly correlated to its relative strength ϵ and the data only approximately determine $\epsilon^{1/3}p_2$. All fits produce a smooth $n(p)$ which are primarily differentiated by their extension, i.e. by the rms momentum. For He we could not reproduce the small starting value $\epsilon = 0.003$ and just put it to 0. Table I assembles starting values and fitted parameters. The shaded area in Fig. 1 shows the extremes for $n^{Fe}(p)$, corresponding to the entries in the Table. The drawn line represents the MD for the given starting values. Fig. 2 for He presents only one fit.

We summarize. Encouraged by the successful extraction of single-atom momentum distributions from cross sections for inclusive scattering of neutrons from mono-atomic quantum gases, we attempted such an extraction from nuclei. The feasibility in the case of NR systems rests on knowledge of the atom-atom interaction in the above cases, and of additional rotation-vibration dynamics in the case of di-atomic molecules, etc. This knowledge en-

dominating AL. The most extreme example is that of hard-core interactions, which produce constant σ_q^{tot} . At least part of the FSI are strictly q -independent³⁴. The above implies that in the large Q^2 -plateau of a plot of $\xi(q, y_G)$ nearly q -independent FSI 'contaminate' the AL there.

ables parametrizations of the structure function, incorporating the above information. Fits to data, enable a separation of q -dependent FSI parts from the q -independent asymptotic limit. That AL has a simple dependence on the desired momentum distribution, which is extracted in a truly model-independent manner.

The nuclear case is in many respects much more complex. For one, the composite nature of nucleons cannot be accurately described as in the molecular case. Consequently, one has to invoke some model dependence, e.g. the one implicit in (4). Moreover, there is a real stumbling block in the shape of FSI, which are barely q -dependent. Consequently a free fit to the data does not lead to fits, smooth in the parameters. Only restrained fits determine characteristics on the momentum distributions.

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Figure captions

Fig. 1 Momentum distribution $n^{Fe}(p)$ from fits of $d^2\sigma$ ¹⁵ for parameters in Table I. Shaded areas correspond to entries in Table I with m_i corresponding to different fixed $n(0)$. The drawn curve is for the starting values of the parameters³¹.

Fig. 2 Momentum distribution $n^{He}(p)$ from fits of $d^2\sigma$ ³⁰ for parameters in Table I. The drawn curve is for the starting values of the parameters³¹.

Table I

	Fe		He
Parameters	Starting values	extracted	extracted
$n(0)(\text{in fm}^{-3})$	69.8		80.51
	76.8		
	58.3		
$p_1 \text{ (in fm}^{-1})$	0.802	0.837 ± 0.011	0.821
		0.813 ± 0.009	
		0.797 ± 0.041	
$\epsilon^{1/3}p_2 \text{ (in fm}^{-1})$	0.432	0.406 ± 0.060	
		0.365 ± 0.060	
		0.268 ± 0.036	
$p_2(\text{in fm}^{-1})$	1.390	1.177 ± 0.136	
		1.371 ± 0.206	
		1.235 ± 0.211	
ϵ	0.03	0.041 ± 0.020	0.0
		0.022 ± 0.009	
		0.020 ± 0.010	
$\sqrt{\langle p^2 \rangle_{n(p)}}(\text{in fm}^{-1})$	1.105	1.108 ± 0.035	1.005
		1.105 ± 0.037	
		1.013 ± 0.021	

Extracted parameters for $n(p)$ of the form Eq. (6). For three fixed values of $n^{Fe}(0)$ are entered the width of the dominant Gaussian p_1 , the combination $\epsilon^{1/3} p_2$ of the relative strength ϵ and width of the second one, as well as each separately, and the root mean square momentum. For He we present only the fit for $\epsilon=0$.



